

Sharp Geometric Rigidity of Isometries on Heisenberg Groups

D. V. Isangulova and S. K. Vodopyanov*

Abstract

We prove sharp geometric rigidity estimates for isometries on Heisenberg groups. Our main result asserts that every $(1 + \varepsilon)$ -quasi-isometry on a John domain of the Heisenberg group \mathbb{H}^n , $n > 1$, is close to some isometry up to proximity order $\sqrt{\varepsilon} + \varepsilon$ in the uniform norm, and up to proximity order ε in the L_p^1 -norm. We give examples showing the asymptotic sharpness of our results.¹

1 Introduction

The following question is studied in elasticity theory: what can we say about a global deformation of a rigid body provided that local deformations are small? This question leads to the mathematical problem [14] formulated below.

A deformation is interpreted as a homeomorphism $f: U \rightarrow \mathbb{R}^3$, where U is an open set in \mathbb{R}^3 . The Jacobi matrix $Df(x)$ is assumed to exist almost everywhere. The symmetric matrix $E(x) = \frac{1}{2}((Df(x))^t Df(x) - I)$ determines $Df(x)$ up to an orthogonal matrix. The matrix $E(x)$ is associated to the deformation or strain tensor (see, for example, [18]). The notion of deformation tensor E plays a key role in elasticity theory (see [18] for instance): various full or partial linearization problems there are based on the assumption that the deformation tensor is sufficiently small. How can this assumption affect $f(x)$ itself? It is known that if $E(x) = 0$ almost everywhere on U then f is a rigid motion under the condition of sufficient regularity. If E is small on U in some sense then what is the global difference between f and a rigid motion

*The research was partially supported by the Russian Foundation for Basic Research (Grant 10-01-00662), the State Maintenance Program for Young Russian Scientists and the Leading Scientific Schools of the Russian Federation (Grant NSh 921.2012.1).

¹*Key words and phrases.* Heisenberg group, sub-Riemannian geometry, quasi-isometry, geometric rigidity.

on the entire domain? If the difference is small globally then this property is called geometric rigidity of isometries.

If f is a homeomorphism with small $E(x)$ then f is locally bi-Lipschitz (see [23] for instance). This leads to a natural interpretation of deformations as bi-Lipschitz mappings.

In 1961 F. John studied this question in a more general setting; namely, he considered a mapping $f: U \rightarrow \mathbb{R}^n$, where U is an open set in \mathbb{R}^n . He showed that *for a locally $(1 + \varepsilon)$ -bi-Lipschitz mapping f , where $\varepsilon < 1$, there exists a motion φ satisfying*

$$\|Df - D\varphi\|_{p,U} \leq C_1 p \varepsilon |U|^{1/p} \quad (1)$$

and

$$\sup_{x \in U} |f(x) - \varphi(x)| \leq C_2 \operatorname{diam}(U) \varepsilon. \quad (2)$$

F. John established (2) for a domain U of a special kind, now called a John domain, and (1) on cubes. Later Yu. G. Reshetnyak [23] established (1) and (2) on John domains without constraints on ε using a different method.

John also studied the question of geometric rigidity under small integral deviations of the deformation tensor [15]: *if U is a cube, $f: U \rightarrow \mathbb{R}^n$ is a mapping of class C^1 , and $\sup |E(x)|$ on U is less than a fixed number then there exists a motion φ such that*

$$\|Df - D\varphi\|_{p,U} \leq C_3 \|E\|_{p,U} \quad \text{if } p > 1 \quad (3)$$

and

$$\sup_{x \in U} |f(x) - \varphi(x)| \leq C_4 \operatorname{diam}(U) \|E\|_{p,U} \quad \text{if } p > n.$$

Recently [6] Friesecke, James, and Müller have demonstrated that (3) holds for every Sobolev mapping of class W_p^1 on a Lipschitz domain U without constraints on

$$\sup_{x \in U} |E(x)| = \sup_{x \in U} \operatorname{dist}(Df(x), \operatorname{SO}(n)).$$

Note that the geometric rigidity problem has a much wider interpretation. The problem can be formulated on any manifold with a notion of differential whose tangent space carries an action of a “model” isometry group.

In this article, we study the geometric rigidity problem on the Heisenberg groups \mathbb{H}^n , $n > 1$. Here is the main result.

Theorem 1. *Consider a John domain U with inner radius α and outer radius β in the Heisenberg group \mathbb{H}^n , $n > 1$. Then, for every $f \in I(1 + \varepsilon, U)$ there exists an isometry θ with*

$$\int_U \exp\left(\left(\frac{\beta}{\alpha}\right)^{2n+3} \frac{N_1 |D_h f(x) - D_h \theta(x)|}{\varepsilon}\right) dx \leq 16|U|$$

and

$$\sup_{x \in U} d(f(x), \theta(x)) \leq N_2 \frac{\beta^2}{\alpha} (\sqrt{\varepsilon} + \varepsilon).$$

Here the constants N_1 and N_2 depend only on n .

Here $I(1+\varepsilon, U)$ is the class of quasi-isometries (see Definition 3), $D_h f(x) = \{X_i f_j(x)\}_{i,j=1,\dots,2n}$ is the approximate horizontal differential, and d is the Carnot–Carathéodory metric.

The dilation $\delta_{1+\varepsilon}$ shows that the proximity orders in Theorem 1 are asymptotically sharp.

D. Morbidelli and N. Arcozzi [1] investigated the geometric rigidity problem for locally bi-Lipschitz mappings of the Heisenberg group \mathbb{H}^1 . We should note, however, that the proximity orders ($\varepsilon^{2^{-11}}$ in the uniform norm and $\varepsilon^{2^{-12}}$ in the Sobolev norm) obtained in [1] are obviously far from being optimal.

Our proof of Theorem 1 develops Reshetnyak’s approach to the subject in the Euclidean case [23]. The proof essentially consists in linearizing the deformation tensor E on Heisenberg groups as a first-order differential operator with constant coefficients whose kernel “almost” coincides with the Lie algebra of the isometry group.

The most important motivation for the study of isometries in sub-Riemannian geometry is given by the recently constructed visualization model (see the papers by G. Citti and A. Sarti [4] and R. K. Hladky and S. D. Pauls [9]). The geometry of the model is based on the roto-translation group, which is a three-dimensional non-nilpotent Lie group. However, it is a contact manifold whose tangent cone at each of point is the Heisenberg group \mathbb{H}^1 . The geometric rigidity problem finds an unexpected interpretation in sub-Riemannian geometry: a local distortion of an image does not incur a loss of global information about it.

In Section 2 we define quasi-isometries on Carnot–Carathéodory spaces, introduce the main concepts used and prove that the class of quasi-isometries under consideration includes locally bi-Lipschitz mappings. In Section 3 we introduce an operator Q linearizing the strain tensor E on the Heisenberg group, and investigate its properties: we describe its kernel and construct a projection onto it. In Section 4 we prove the geometric rigidity of isometries on the balls contained in a given domain. In Section 5 we prove Theorem 1 on a John domain. There we also obtain a partial extension of Theorem 1 to a Hölder domain. In the Appendix we prove some auxiliary results.

The main results of this article were announced in [28].

2 Quasi-isometries

Definition 1 (cf. [8, 16, 20]). Fix a connected Romanian C^∞ -manifold \mathbb{M} of topological dimension N . The manifold \mathbb{M} is called a *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \dots \subsetneq H_i\mathbb{M} \subsetneq \dots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that each point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of $C^{1,\alpha}$ -smooth vector fields X_1, \dots, X_N , $\alpha \in (0, 1]$, enjoying the following two properties.

For each $v \in U$,

(1) $H_i\mathbb{M}(v) = H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$ is a subspace of $T_v\mathbb{M}$ of a constant dimension $\dim H_i$, $i = 1, \dots, M$;

(2) we have

$$[X_i, X_j](v) = \sum_{k: \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v) \quad (4)$$

where the *degree* $\deg X_k$ is defined as $\min\{m \mid X_k \in H_m\}$;

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the *Carnot manifold*:

(3) the quotient mapping $[\cdot, \cdot]_0 : H_1 \times H_j/H_{j-1} \mapsto H_{j+1}/H_j$ induced by the Lie bracket is an epimorphism for all $1 \leq j < M$. Here $H_0 = \{0\}$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold \mathbb{M} .

The intrinsic *Carnot–Carathéodory distance* d between two points $x, y \in \mathbb{M}$ is defined as the infimum of lengths of the horizontal curves joining x and y (a piecewise smooth curve γ is horizontal if $\dot{\gamma}(t) \in H\mathbb{M}(\gamma(t))$). This distance is correctly-defined [16] and non-Riemannian if $n = \dim H\mathbb{M} \neq N$.

Let U be a domain in \mathbb{M} and $\{X_1, \dots, X_n\}$ be an orthonormal basis of $H\mathbb{M}$ on U from Definition 1. The Sobolev space $W_q^1(U)$, $1 \leq q \leq \infty$, consists of the functions $f: U \rightarrow \mathbb{R}$ possessing the weak derivative $X_i f$ along the vector field X_i , $i = 1, \dots, n$, and having a finite norm

$$\|f\|_{W_q^1(U)} = \|f\|_{q,U} + \|\nabla_{\mathcal{L}} f\|_{q,U},$$

where $\nabla_{\mathcal{L}} f = (X_1 f, \dots, X_n f)$ is the *subgradient* of f and $\|\cdot\|_{q,U}$ stands for the L_q -norm of a measurable function on U . Recall that a locally integrable function $g_i : U \rightarrow \mathbb{R}$ is called the

weak derivative of a function f along the vector field X_i if $\int_U g_i \psi dx = - \int_U f X_i \psi dx$ for every test function $\psi \in C_0^\infty(U)$.

If $f \in W_q^1(U)$ for every bounded open set U , with $\overline{U} \subset \Omega$, then f is said to be of class $W_{q,\text{loc}}^1(\Omega)$.

Definition 2. A mapping $f : \Omega \rightarrow \mathbb{M}$ belongs to the Sobolev class $W_{q,\text{loc}}^1(\Omega, \mathbb{M})$ if the following properties hold:

- (A) for each $z \in \mathbb{M}$ the function $[f]_z : x \in \Omega \mapsto d(f(x), z)$ belongs to $W_{q,\text{loc}}^1(\Omega)$;
- (B) the family of functions $\{\nabla_{\mathcal{L}}[f]_z\}_{z \in \mathbb{M}}$ has a majorant in $L_{q,\text{loc}}(\Omega)$: there exists a function $g \in L_{q,\text{loc}}(\Omega)$ independent of z such that $|\nabla_{\mathcal{L}}[f]_z(x)| \leq g(x)$ for almost all $x \in \Omega$.

If f is a Sobolev mapping then it can be redefined on a set of measure zero to be absolutely continuous on almost all lines of the horizontal vector fields. In this case there exist derivatives $X_i f(x)$ a. e. in Ω , moreover $X_i f(x) \in H_{f(x)}\mathbb{M}$, $i = 1, \dots, n$ (see [21] in Carnot groups, and [27] in Carnot–Carathéodory spaces). A transformation of the basis vectors $X_i(x)$, $i = 1, \dots, n$, of the horizontal subspace $H_x\mathbb{M}$ into the horizontal vectors $(X_i f)(x) \in H_{f(x)}\mathbb{M}$ determines a mapping $D_h f(x)$ from the horizontal space $H_x\mathbb{M}$ into $H_{f(x)}\mathbb{M}$ for almost all $x \in \Omega$, which is called the *approximate horizontal differential*.

The mapping $D_h f$ in turn generates almost everywhere a morphism Df of graded Lie algebras [27]. The determinant of the matrix $Df(x)$ is called the *Jacobian* of f and is denoted by $J(x, f)$.

Definition 3. Let U be an open set in a Carnot–Carathéodory space \mathbb{M} , and let $f : U \rightarrow \mathbb{M}$ be a nonconstant mapping of Sobolev class $W_{1,\text{loc}}^1(U, \mathbb{M})$. The mapping f belongs to the class $I(L, U)$, $L \geq 1$, if $J(x, f)$ keeps its sign on U and $L^{-1}|\xi| \leq |D_h f(x)\xi| \leq L|\xi|$ for all $\xi \in H\mathbb{M}(x)$ and almost every $x \in U$.

Obviously, a quasi-isometric mapping belongs to the Sobolev space $W_{p,\text{loc}}^1$ for all $p \geq 1$.

Recall that a mapping $f : U \rightarrow \mathbb{M}$ is *locally L -Lipschitz* if every point $x \in U$ has a neighborhood V with $\overline{V} \subset U$ such that the inequality $d(f(y), f(z)) \leq Ld(y, z)$ is valid for all $y, z \in V$; also, f is *locally L -bi-Lipschitz* if $\frac{1}{L}d(y, z) \leq d(f(y), f(z)) \leq Ld(y, z)$ for all $y, z \in V$.

Lemma 1. *If f belongs to $I(L, U)$ then f is locally L -Lipschitz. If in addition f is a local homeomorphism then f is locally L -bi-Lipschitz.*

Conversely, every locally L -bi-Lipschitz mapping of an open set U belongs to $I(L, U)$.

Proof. Since for every horizontal curve $\gamma: [0, T] \rightarrow U$ the curve $f(\gamma)$ is also horizontal, it suffices to prove that

$$l(f(\gamma)) \leq Ll(\gamma). \quad (5)$$

If $f \circ \gamma \in ACL$ and $D_h f(\gamma(t))$ is defined for almost all t then (5) is obvious:

$$l(f(\gamma)) = \int_0^T \left| \frac{d}{dt} f(\gamma(t)) \right| dt = \int_0^T |D_h f(\gamma(t))| \left| \frac{d}{dt} \gamma(t) \right| dt \leq Ll(\gamma). \quad (6)$$

Take a point $a \in U$ and a field $X \in H\mathbb{M}$. Consider the curve $\gamma = \exp(tX)(a)$, $\gamma: [0, T] \rightarrow U$, and a surface S transversal to X at a such that the foliation $\Phi = \{\exp(tX)(x), x \in S\}$ lies in U . For the function $[f]_z(x) = d(z, f(x))$ there exists a function $g \in L_1$ independent of z and satisfying both $|[f]_z(x) - [f]_z(y)| \leq d(x, y)(g(x) + g(y))$ and $|\nabla_{\mathcal{L}}[f]_z(x)| \leq Mg(x)$. By Fubini's theorem, g belongs to the class L_1 for almost all curves of the foliation Φ . Consequently, $|[f]_z(x) - [f]_z(y)| \leq M \int_{[x, y]} g dt$ on each of these curves. Choosing z arbitrarily close to $f(y)$, we infer that $d(f(x), f(y)) \leq M \int_{[x, y]} g dt$ and, hence, $f \in ACL$ and is differentiable almost everywhere on almost all curves in Φ . Consequently, (6) holds on almost all curves in Φ . Choose a sequence of curves $\gamma_n \in \Phi$ converging to γ and satisfying (6). Since f is continuous, $f \circ \gamma_n \rightarrow f \circ \gamma$ pointwise as $n \rightarrow \infty$. The lower semicontinuity of length yields

$$l(f \circ \gamma) \leq \liminf_{n \rightarrow \infty} l(f \circ \gamma_n) \leq \liminf_{n \rightarrow \infty} Ll(\gamma_n) = Ll(\gamma).$$

Thus, the curve $\gamma = \exp(tX)(a)$ satisfies (5).

Consider a domain V with $\overline{V} \subset U$. Fix two points $x, y \in V$. Then the points x and y can be joined by a piecewise smooth horizontal curve γ in U consisting of pieces of integral curves of horizontal vector fields X_i , $i = 1, \dots, n$. Moreover, $l(\gamma) \leq c d(x, y)$ with $c \geq 1$ dependent on V [16, Proof of Theorem 2.8.4]; hence, $l(f(\gamma)) \leq cLl(\gamma)$. Thus, f is locally Lipschitz with the Lipschitz constant cL and $l(f \circ \gamma) \leq cLl(\gamma)$ for any horizontal curve in V . Verify that f is locally L -Lipschitz.

Suppose now that $\gamma: [0, T] \rightarrow V$ is a horizontal curve parametrized by arc length. Put $\Sigma = \{t \in [0, T] \mid \gamma \text{ is not differentiable at } t\}$. Then $|\Sigma| = 0$, $|\dot{\gamma}(t)| = 1$ for all $t \in [0, T] \setminus \Sigma$ and

$$d(\gamma(t+s), \exp(s\dot{\gamma}(t))(\gamma(t))) = o(s) \quad \text{as } s \rightarrow 0 \quad \text{for all } t \in [0, T] \setminus \Sigma.$$

Consider arbitrary $\varepsilon > 0$ and $\delta > 0$. Now we construst a partition of the interval $[0, T]$ by intervals with diameter less than δ .

First, we cover Σ by open intervals $\{W\}_{W \in \mathcal{W}}$ centered at $[0, T]$ such that $|W| < \delta$, $\Sigma \subset \bigcup_{W \in \mathcal{W}} W$ and $\sum_{W \in \mathcal{W}} |W| < \varepsilon$. Second, we cover set $[0, T] \setminus \Sigma$ by intervals $\mathcal{U} = \{(t - \delta(t), t +$

$\delta(t)) : t \in [0, T] \setminus \Sigma\}$ where $\delta(t) > 0$ satisfies

$$d(\gamma(t+s), \exp(s\dot{\gamma}(t))(\gamma(t))) < \varepsilon s \quad \text{for all } s \leq \delta(t). \quad (7)$$

Without loss of generality we may assume that $\delta(t) < \delta/2$.

Since the set $[0, T]$ is compact, there is a finite covering of $[0, T]$ by open intervals $\{U_i\}$ with $U_i \in \mathcal{U}$ or $U_i \in \mathcal{W}$. By Lemma 2, there is a partition of $[0, T]$ by intervals $P_k = [t_k, t_{k+1})$ with the following properties: $P_k \subset U_i$, for some i , and $\overline{P_k}$ contains the center of U_i . The latter we denote by τ_k . Obviously, $t_{k+1} - t_k < \delta$. Divide indices into two groups: $k \in I$ if $P_k \subset U_i$, $U_i \in \mathcal{U}$; and $k \in J$ if $P_k \in U_j$, $U_j \in \mathcal{W}$.

Since γ is parameterized by arc length it follows that

$$\sum_{k \in J} d(f(\gamma(t_k)), f(\gamma(t_{k+1}))) \leq \sum_{k \in J} l(f \circ \gamma|_{P_k}) \leq cL \sum_{k \in J} l(\gamma|_{P_k}) \leq cL \sum_{W \in \mathcal{W}} |W| \leq cL\varepsilon.$$

For $k \in I$ we set

$$\sigma_k(t) = \exp((t - \tau_k)\dot{\gamma}(\tau_k))(\gamma(\tau_k)), \quad t \in [t_k, t_{k+1}].$$

Applying $|\dot{\sigma}_k(t)| = |\dot{\gamma}(\tau_k)| = 1$ we obtain $l(\sigma_k) = t_{k+1} - t_k$ and $\sum_{k \in I} l(\sigma_k) \leq T = l(\gamma)$. Relation (7) yields

$$\sum_{k \in I} d(\sigma_k(t_k), \gamma(t_k)) + d(\sigma_k(t_{k+1}), \gamma(t_{k+1})) < \sum_{k \in I} \varepsilon(t_{k+1} - t_k) \leq \varepsilon T.$$

Therefore

$$\begin{aligned} & \sum_{k \in I} d(f(\gamma(t_k)), f(\gamma(t_{k+1}))) \\ & \leq \sum_{k \in I} d(f(\gamma(t_k)), f(\sigma_k(t_k))) + d(f(\sigma_k(t_{k+1})), f(\gamma(t_{k+1}))) + l(f \circ \sigma_k) \\ & \leq \sum_{k \in I} cL(d(\gamma(t_k), \sigma_k(t_k)) + d(\gamma(t_{k+1}), \sigma_k(t_{k+1}))) + Ll(\sigma_k) \leq cL\varepsilon T + Ll(\gamma). \end{aligned}$$

Finally,

$$l(f \circ \gamma) = \lim_{\delta \rightarrow 0} \sum_{k \in I \cup J} d(f(\gamma(t_k)), f(\gamma(t_{k+1}))) \leq Ll(\gamma) + cL\varepsilon T + cL\varepsilon.$$

Since ε is arbitrary, it follows that $l(f \circ \gamma) \leq Ll(\gamma)$.

The converse is obvious. □

The following partition lemma was used in the proof of Lemma 1. Proof of this lemma is based on the induction method.

Lemma 2. *Consider the finite covering of a closed segment $[a, b]$ by open intervals $\{U_i\}$. Suppose $x_i \in [a, b]$ where x_i is centre of interval U_i . Then there is a partition of $[a, b]$ by intervals $\{P_k\}$ satisfying $x_i \in \overline{P_k} \subset U_i$ for some i .*

THE HEISENBERG GROUP. The Heisenberg group \mathbb{H}^n is an example of homogeneous Carnot manifold. We may identify the points of \mathbb{H}^n with the points of \mathbb{R}^{2n+1} . The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2x_{i+n} \frac{\partial}{\partial x_{2n+1}}, \quad X_{i+n} = \frac{\partial}{\partial x_{i+n}} - 2x_i \frac{\partial}{\partial x_{2n+1}}, \quad i = 1, \dots, n,$$

constitute a basis of the horizontal subbundle $H\mathbb{H}^n$.

Together with the vector field $X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}$ they constitute the standard basis of the Lie algebra. The only nontrivial commutation relations are

$$[X_j, X_{j+n}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

From now on we consider the Heisenberg group. It is convenient to use the complex notation: a point $x \in \mathbb{H}^n$ may be regarded as (z, t) , where

$$z = (x_1 + ix_{n+1}, \dots, x_n + ix_{2n}) \in \mathbb{C}^n \quad \text{and} \quad t = x_{2n+1} \in \mathbb{R}.$$

Then the vector fields

$$Z_j = \frac{1}{2}(X_j - iX_{j+n}) = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{1}{2}(X_j + iX_{j+n}) = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n,$$

$$T = X_{2n+1} = \frac{\partial}{\partial t}$$

constitute a left-invariant basis of the Lie algebra.

The *dilation* δ_s , for $s > 0$, acts on the Heisenberg group as $\delta_s(z, t) = (sz, s^2t)$ and is an automorphism of it. The *homogeneous norm* $\rho(z, t) = (|z|^4 + t^2)^{1/4}$ defines the *Heisenberg metric* ρ as $\rho(x, y) = \rho(x^{-1} \cdot y)$, $x, y \in \mathbb{H}^n$. Observe that the Heisenberg metric is a metric and not just a quasi-metric: $\rho(x \cdot y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{H}^n$ (see [11] for instance). It is also known that the Heisenberg metric ρ and the Carnot–Carathéodory metric d are equivalent: there exists a constant $c > 1$ such that $c^{-1}d(x, y) \leq \rho(x, y) \leq cd(x, y)$ for all $x, y \in \mathbb{H}^n$.

The Lebesgue measure \mathbb{R}^{2n+1} is a bi-invariant Haar measure. For the ball $B(x, r) = \{y \in \mathbb{H}^n : \rho(x, y) < r\}$ we have $|B(x, r)| = r^\nu |B(0, 1)|$, where $\nu = 2n+2$ is the *homogeneous dimension* of the group \mathbb{H}^n .

Consider a Sobolev mapping f . Since Df is a homomorphism of graded Lie algebras, it follows that for almost every $x \in \Omega$ there exists a number $\lambda(x, f)$ such that

$$Df(x)X_{2n+1} = \lambda(x, f)X_{2n+1}.$$

Furthermore [17], $\lambda(x, f)^n = \det D_h f(x)$ and $\lambda(x, f)^{n+1} = J(x, f)$. In particular, $J(x, f) \geq 0$ almost everywhere on Ω for odd n . Consequently, for odd n , there are no Sobolev mappings changing the topological orientation. We now give the definition of orientation introduced by A. Korányi and H. M. Reimann in [17].

Definition 4. A mapping f of the Sobolev class $W_{1,\text{loc}}^1(\Omega, \mathbb{H}^n)$ *preserves (changes) KR-orientation* if $\lambda(x, f) > 0$ ($\lambda(x, f) < 0$) for almost all $x \in \Omega$.

A mapping $f \in I(1, U)$ is called an *isometry* on U . Every isometric mapping of the Heisenberg group \mathbb{H}^n has the form $\pi_a \circ \varphi_A$ or $\iota \circ \pi_a \circ \varphi_A$, where $\iota(z, t) = (\bar{z}, -t)$ is a reflection, $\pi_a(x) = a \cdot x$ with $a \in \mathbb{H}^n$ is a left translation, $\varphi_A(x) = (Az, t)$ with $A \in U(n)$ is a rotation [17]. Isometries preserve distance in the Heisenberg metric as well as in the Carnot–Carathéodory metric. It is also worth noting that $D_h \varphi$ is a constant mapping for every isometry φ .

A quasi-isometric mapping is not only locally Lipschitz but also a mapping with bounded distortion. Consider a domain U in \mathbb{H}^n . Recall that a nonconstant mapping $f: U \rightarrow \mathbb{H}^n$ of the class $W_{\nu,\text{loc}}^1(U, \mathbb{H}^n)$ is called a *mapping with bounded distortion* if there exists a constant $K \geq 1$ such that the approximate horizontal differential satisfies $|D_h f(x)|^\nu \leq K^{n+1} J(x, f)$ for almost every $x \in U$. The smallest constant K in this inequality is called the (*linear*) *distortion coefficient* of f and is denoted by $K(f)$.

Suppose that $f \in I(L, U)$. Denote by λ_1 and λ_0 eigenvalues of $D_h f(x)$ of the largest and smallest absolute values. Clearly, $|\lambda_1| \leq L$, $|\lambda_0| \geq L^{-1}$, and $|D_h f(x)| = |\lambda_1|$. We also have $|J(x, f)| = |\lambda_1 \lambda_0|^{n+1}$. Hence,

$$|D_h f(x)|^{2n+2} = |\lambda_1|^{2n+2} = \left(\frac{|\lambda_1|}{|\lambda_0|} \right)^{n+1} |J(x, f)| \leq L^{2n+2} |J(x, f)|.$$

Thus, if $J(x, f)$ is nonnegative almost everywhere then f is a mapping of bounded distortion with $K(f) = L^2$. If $J(x, f)$ is nonpositive almost everywhere then $\iota \circ f$ is a mapping with bounded distortion with $K(\iota \circ f) = L^2$.

3 The Operator Q

In this section we introduce a differential operator Q approximating the equation $(D_h f(x))^t D_h f(x) = I$ to first order. This equation means that $D_h f(x)$ is an orthogonal matrix. In contrast to the Euclidean case, the horizontal differential of a Sobolev mapping has some additional structure: up to a factor, $D_h f(x)$ is a symplectic matrix. Therefore, the operator Q consists of two parts: the first is responsible for orthogonality, and the second, for symplecticity.

3.1 The main lemma for the operator Q

Given a domain U in \mathbb{H}^n , denote by Q the homogeneous differential operator acting on a mapping $u: U \rightarrow \mathbb{R}^{2n}$ as

$$Qu = \frac{1}{2} \begin{pmatrix} D_h u + (D_h u)^t \\ D_h u + J D_h u J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (8)$$

Here the $2n \times 2n$ matrix $D_h u$ equals $(X_i u_j)_{i,j=1,\dots,2n}$. The operator Q also acts on mappings u from U to \mathbb{H}^n . In this case, $D_h u$ in (8) stands for the approximate horizontal differential of u .

In complex notation, the operator Q is defined as

$$Qu = \begin{pmatrix} \frac{1}{2}(Zu + (Zu)^*) \\ \overline{Z}u \end{pmatrix}, \quad u: U \rightarrow \mathbb{C}^n.$$

The following lemma expresses the main inequality for the operator Q :

Lemma 3. *Given an open set U in \mathbb{H}^n and a mapping f of class $I(L, U)$ preserving KR -orientation, the inequality*

$$|Q(x^{-1} \cdot f(x))| \leq \frac{L^2 - 1}{2} (|D_h f(x) - I| + 2) + \frac{1}{2} |D_h f(x) - I|^2 \quad (9)$$

holds almost everywhere on U .

Proof. Put $x^{-1} \cdot f(x) = u(x)$. Then $D_h f(x) = D_h u(x) + I$ for almost all $x \in U$. We have

$$(D_h f(x))^t D_h f(x) = I + (D_h u(x))^t + D_h u(x) + (D_h u(x))^t D_h u(x).$$

Hence,

$$2Q_1 u(x) = (D_h f(x))^t D_h f(x) - I - (D_h u(x))^t D_h u(x),$$

where $Q_1 u(x) = \frac{1}{2}((D_h u(x))^t + D_h u(x))$ is a first-order differential operator with constant coefficients. The relation $|(D_h f(x))^t D_h f(x) - I| \leq L^2 - 1$ yields

$$|Q_1 u(x)| \leq \frac{L^2 - 1}{2} + \frac{1}{2} |D_h u(x)|^2.$$

It is easy to verify that $|\overline{Z}f| = |\frac{1}{2}(D_h f + J D_h f J)|$ and $|Zf| = |\frac{1}{2}(D_h f - J D_h f J)| \leq |D_h f|$. Since f preserves KR -orientation and is a mapping with bounded distortion, the Beltrami system [17, Theorem C] implies that

$$|\overline{Z}u| = |\overline{Z}f| \leq \frac{K-1}{K+1} |Zf| \leq \frac{K-1}{K+1} (|D_h f - I| + 1) \leq \frac{L^2 - 1}{2} (|D_h u| + 1).$$

It remains to observe that $|Qu| \leq |Q_1 u| + |\frac{1}{2}(D_h u + J D_h u J)|$. □

3.2 The kernel of the operator Q

To prove the main results of this paper, we apply the coercive estimates for Q in (9). On general Carnot groups, Isangulova and Vodopyanov established coercive estimates for homogeneous differential operators with constant coefficients and finite-dimensional kernels [13]. On Heisenberg groups, Romanovskiĭ obtained this result earlier in [24, 25]. To apply the coercive estimates, we only have to show that the kernel of Q is finite-dimensional.

Lemma 4. *The kernel of the operator Q on the Sobolev class $W_{p,\text{loc}}^1(\mathbb{H}^n, \mathbb{C}^n)$, $p > 1$, is finite-dimensional: $u \in \ker Q$ if and only if*

$$\begin{aligned} u(z, t) &= a + Kz, \quad \text{where } a \in \mathbb{C}^n \text{ and } K + K^* = 0 \quad \text{for } n > 1; \\ u(z, t) &= a + ikz + tb + iz^2\bar{b} + i|z|^2b, \quad \text{where } a, b \in \mathbb{C}, k \in \mathbb{R} \quad \text{for } n = 1; \end{aligned}$$

Proof. (I) Take a C^∞ -function $u: \mathbb{H}^n \rightarrow \mathbb{C}^n$ in the kernel of Q . In complex notation, $u \in \ker Q$ if and only if

$$Zu = -(Zu)^*, \quad \bar{Z}u = Z\bar{u} = 0.$$

If u is independent of $t = x_{2n+1}$ then it is easy to see that

$$u(z, t) = a + Kz, \quad \text{where } z \in \mathbb{C}^n, t \in \mathbb{C}, a \in \mathbb{C}^n, \text{ and } K + K^* = 0.$$

Suppose that u depends on $t = x_{2n+1}$. We have

$$-2iZ_m T u_k = Z_m(Z_k \bar{Z}_k - \bar{Z}_k Z_k)u_k = -Z_m \bar{Z}_k Z_k u_k = Z_m \bar{Z}_k \bar{Z}_k \bar{u}_k = \bar{Z}_k \bar{Z}_k Z_m \bar{u}_k = 0$$

for all $m \neq k$. If $m = k$ then

$$-2iZ_k T u_k = Z_k(Z_j \bar{Z}_j - \bar{Z}_j Z_j)u_k = -Z_k \bar{Z}_j Z_j u_k = Z_k \bar{Z}_j \bar{Z}_k \bar{u}_j = -2i\bar{Z}_j T \bar{u}_j = 2iZ_j T u_j,$$

where $j \neq k$. Thus, $Z_k T u_k = 0$ for all $k = 1, \dots, n$ provided that $n > 2$.

(II) Consider the case $n > 2$. We have $Tu = \lambda$ with $\lambda \in \mathbb{C}^n$. Verify that $\lambda = 0$. We have

$$u = a + Kz + \lambda(t + i|z|^2) + P(z)$$

with $a, \lambda \in \mathbb{C}^n$ and $K + K^* = 0$. Here $P = (P_1, P_2, \dots, P_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$, where $P_k(z) = \sum_{l,s=1}^n p_{ls}^k z_l z_s$, $k = 1, \dots, n$, are polynomials of degree 2 depending only on z , $p_{ls}^k = p_{sl}^k$. Here we consider the function $t + i|z|^2$ since its differential along \bar{Z}_k vanishes for all $k = 1, \dots, n$.

Hence,

$$Z_k u_l = K_{lk} + 2i\bar{z}_k \lambda_l + \sum_{j=1}^n (p_{kj}^l z_j + p_{jk}^l z_j) = K_{lk} + 2i\bar{z}_k \lambda_l + 2 \sum_{j=1}^n p_{kj}^l z_j,$$

$$\overline{Z_l u_k} = \overline{K_{kl}} - 2iz_l \overline{\lambda_k} + 2 \sum_{j=1}^n \overline{p_{lj}^k} \overline{z_j}.$$

The coefficients of $\overline{z_k}$ and z_l in the equation $Z_k u_l + \overline{Z_l u_k} = 0$ are

$$2i\lambda_l + 2\overline{p_{lk}^k} = 0, \quad -2i\overline{\lambda_k} + 2p_{kl}^l = 0, \quad p_{ls}^k = 0 \text{ for all } s \neq k.$$

Since $p_{ls}^k = p_{sl}^k$, we infer that $P_k = 0$ and $\lambda_k = 0$ for all k .

(III) Consider the case $n = 2$. We have

$$Z_2 T u_1 = Z_1 T u_2 = 0 \quad \text{and} \quad Z_1 T u_1 = -\overline{Z_1} T \overline{u_1} = -Z_2 T u_2 = \overline{Z_2} T \overline{u_2} = \mu.$$

The following relations show that μ is a constant:

$$\begin{aligned} Z_1 \mu &= Z_1 \overline{Z_2} T \overline{u_2} = 0, & \overline{Z_1} \mu &= -\overline{Z_1} Z_2 T u_2 = 0, \\ Z_2 \mu &= -Z_2 \overline{Z_1} T \overline{u_1} = 0, & \overline{Z_2} \mu &= \overline{Z_2} Z_1 T u_1 = 0. \end{aligned}$$

Hence, $T u_1 = \lambda_1 + \mu z_1$ and $T u_2 = \lambda_2 - \mu z_2$. Thus,

$$u_1 = (t + i|z|^2)(\lambda_1 + \mu z_1) + P_1(z), \quad u_2 = (t + i|z|^2)(\lambda_2 - \mu z_2) + P_2(z).$$

Here we write down u up to the known term $a + Kz$ and $P_k = a_k z_1^2 + b_k z_1 z_2 + c_k z_2^2$, $k = 1, 2$, are polynomials of degree 2 depending only on z_1, z_2 .

It follows that

$$\begin{aligned} 0 = Z_1 u_1 + \overline{Z_1 u_1} &= 2i\overline{z_1}(\lambda_1 + \mu z_1) + (t + i|z|^2)\mu + 2a_1 z_1 + b_1 z_2 \\ &\quad - 2iz_1(\overline{\lambda_1} + \overline{\mu} \overline{z_1}) + (t - i|z|^2)\overline{\mu} + 2\overline{a_1} \overline{z_1} + \overline{b_1} \overline{z_2}. \end{aligned}$$

The coefficients of $|z|^2$ and t are equal to $i\mu - i\overline{\mu}$ and $\mu + \overline{\mu}$ respectively. Thus, $\mu = 0$. Clearly, $b_1 = 0$ and $a_1 = i\overline{\lambda_1}$. Similarly, $b_2 = 0$ and $c_2 = i\overline{\lambda_1}$. The equality

$$Z_2 u_1 = 2i\overline{z_2} \lambda_1 + 2c_1 z_2 = -\overline{Z_1 u_2} = -2iz_1 \overline{\lambda_2} - 2\overline{a_2} \overline{z_1}$$

implies that $\lambda_1 = \lambda_2 = P_1 = P_2 = 0$.

(IV) Consider the case $n = 1$. A mapping $u = (u_1, u_2): \mathbb{H}^1 \rightarrow \mathbb{R}^2$ belongs to $\ker Q$ if and only if

$$X u_1 = 0, \quad Y u_2 = 0, \quad Y u_1 + X u_2 = 0.$$

Put $\varphi = Y u_1 = -X u_2$. It satisfies

$$\begin{aligned} X^2 \varphi &= X^2 Y u_1 = X X Y u_1 - X Y X u_1 = -4X T u_1 = -4T X u_1 = 0, \\ Y^2 \varphi &= -Y^2 X u_2 = Y X Y u_2 - Y Y X u_2 = -4Y T u_2 = -4T Y u_2 = 0, \\ Y X \varphi + X Y \varphi &= Y X Y u_1 - X Y X u_2 \\ &= Y(-4T + YX)u_1 + X(-4T - XY)u_2 = -4T(Y u_1 + X u_2) = 0. \end{aligned}$$

Verify that $T\varphi \equiv \text{const.}$ We have

$$-4XT\varphi = X(XY\varphi - YX\varphi) = -2XYX\varphi = (XY - YX)X\varphi = XYX\varphi$$

and

$$-4YT\varphi = Y(XY\varphi - YX\varphi) = 2YXY\varphi = (XY - YX)Y\varphi = -YXY\varphi.$$

Hence, $XYX\varphi = YXY\varphi = 0$ and $XT\varphi = YT\varphi = 0$. Thus, $T\varphi = \lambda \in \mathbb{R}$ and $\varphi = \lambda t + \psi(x, y)$. Since $X^2\varphi = \frac{\partial^2\psi}{\partial x^2} = 0$, $Y^2\varphi = \frac{\partial^2\psi}{\partial y^2} = 0$, and $XY\varphi + YX\varphi = 2\frac{\partial^2\psi}{\partial y\partial x} = 0$, we conclude that ψ is a linear function of x and y .

Thus, $\varphi = \alpha + \lambda t + \mu x + \nu y$. It remains to calculate u_1 and u_2 . The systems

$$\begin{cases} Xu_1 = 0, \\ Yu_1 = \varphi, \end{cases} \quad \begin{cases} Xu_2 = -\varphi, \\ Yu_2 = 0 \end{cases}$$

yield

$$u_1 = c_1 + \alpha y - \frac{\mu}{4}t + \frac{\mu xy + \nu y^2}{2}, \quad u_2 = c_2 - \alpha x - \frac{\nu}{4}t - \frac{\nu xy + \mu x^2}{2}.$$

(v) Consider a mapping u of Sobolev class $W_{p,\text{loc}}^1(\mathbb{H}^n, \mathbb{R}^{2n})$ satisfying $Qu = 0$ in the sense of distributions. We show that $u \in \ker Q$, where $\ker Q$ is the finite-dimensional space found in the smooth case. Consider a ball B in \mathbb{H}^n and construct a sequence $u_k \in C^\infty(\mathbb{H}^n, \mathbb{R}^{2n})$ such that $\|u - u_k\|_{W_p^1(B)} \rightarrow 0$ as $k \rightarrow \infty$. We showed above that the kernel of Q is finite-dimensional on smooth mappings. Hence, by Theorem 1 of [13], there exists a projection P onto $\ker Q$ such that

$$\|u_k - Pu_k\|_{W_p^1(B)} \leq C\|Qu_k\|_{p,B}.$$

Passing to the limit as $k \rightarrow \infty$, we infer that $\|u - Pu\|_{W_p^1(B)} \leq C\|Qu\|_{p,B} = 0$, where $Pu = \lim_{k \rightarrow \infty} Pu_k$. Since $Pu_k \in \ker Q$, it follows that Pu also belongs to $\ker Q$. Finally, $u = Pu \in \ker Q$. \square

3.3 Projection onto the Kernel of the Operator Q

In this subsection, we construct a projection onto $\ker Q$ convenient for further calculations.

Put

$$\text{Box}(a, r) = \{ay \in \mathbb{H}^n : y = (y_1, \dots, y_{2n+1}), |y_i| < r, i = 1, \dots, 2n, |y_{2n+1}| < r^2\}.$$

It is easy to verify that

$$\text{Box}(a, \varkappa r) \subset B(a, r) \subset \text{Box}(a, r), \quad \text{where } \varkappa = (4n^2 + 1)^{-1/4},$$

$$|\text{Box}(a, r)| = 2^{2n+1}r^\nu, \quad \int_{\text{Box}(0, r)} |z_i|^2 dx = \frac{2^\nu r^{\nu+2}}{3} \quad \text{for all } i = 1, \dots, n.$$

By [13], we have the following result: *given a ball $B \subset \mathbb{H}^n$, $n > 1$, and $p > 1$ there is a projection Π from $W_p^1(B, \mathbb{R}^{2n})$ onto the kernel of Q such that*

$$\|f - \Pi f\|_{W_p^1(B)} \leq C \|Qf\|_{p, B} \quad \text{for every } f \in W_p^1(B, \mathbb{R}^{2n}).$$

By analogy with Theorem 3.2 of Chapter 3 of [23], we can show that the coercive estimates hold for every projection onto the kernel of Q .

Proposition 1 ([12, Proposition 2]). *Consider a ball B on the Heisenberg group \mathbb{H}^n , $n > 1$, $p > 1$, and a projection P from $W_p^1(B, \mathbb{R}^{2n})$ onto $\ker(Q)$. Then there is a constant $C > 0$ such that*

$$\|u - Pu\|_{W_p^1(B)} \leq C \|Qu\|_{p, B}$$

for every $u \in W_p^1(B, \mathbb{R}^{2n})$.

We construct a projection P from $W_p^1(B(0, \frac{3}{10}), \mathbb{C}^n)$ for $B(0, \frac{3}{10}) \subset \mathbb{H}^n$ with $n > 1$ and $p > 1$, onto the kernel of Q .

Consider the complex-valued $n \times n$ matrix $A(u)$,

$$[A(u)]_{ij} = \frac{2^{\nu+4}3}{\varkappa^{\nu+2}} \int_{\text{Box}(0, \frac{\varkappa}{4})} u_i(x) \bar{z}_j dx, \quad i, j = 1, \dots, n;$$

and the vector $a(u) \in \mathbb{C}^n$,

$$[a(u)]_i = \frac{2^{\nu+1}}{\varkappa^\nu} \int_{\text{Box}(0, \frac{\varkappa}{4})} u_i(x) dx, \quad i = 1, \dots, n.$$

The following properties are obvious:

- (1) if $u \equiv \text{const}$ then $a(u) = u$ and $A(u) = 0$;
- (2) if $u(z, t) \equiv z$ then $a(u) = 0$ and $A(u) = I$;
- (3) if $B \in U(n)$ then $a(Bu) = Ba(u)$ and $A(Bu) = BA(u)$.

Definition 5. Define the projection P onto the kernel of Q as

$$Pu = K(u)z + a(u)$$

for $u \in W_p^1(B(0, \frac{3}{10}), \mathbb{C}^n)$, where $K(u) = \frac{A(u) - [A(u)]^*}{2}$ is a skew-Hermitian $n \times n$ matrix.

Lemma 5. *Suppose that $\varepsilon < \sqrt{\frac{2}{n}} \left(\frac{2}{\varkappa}\right)^{n+1}$. If $u \in W_2^1(B(0, \frac{3}{10}), \mathbb{C}^n)$ satisfies $|u(x) - z| \leq \varepsilon$ for all $x = (z, t) \in B(0, \frac{3}{10})$ then there is a unitary $n \times n$ matrix $V \in U(n)$ such that $|V - I| < \frac{n\varkappa^{n+1}}{2^n} \varepsilon$ and $P(Vu) \equiv \text{const}$.*

Proof. Put $A = A(u)$. Given a vector $\xi \in \mathbb{C}^n$, we have

$$\begin{aligned} |[A - I]\xi|^2 &= |[A(u) - A(z)]\xi|^2 = \frac{2^{\nu+4}3}{\varkappa^{\nu+2}} \sum_{i=1}^n \left| \int_{\text{Box}(0, \frac{\varkappa}{4})} \sum_{j=1}^n (u_i(x) - z_i) \bar{z}_j \xi_j dx \right|^2 \\ &\leq \frac{2^{\nu+4}3}{\varkappa^{\nu+2}} \int_{\text{Box}(0, \frac{\varkappa}{4})} |u(x) - z|^2 dx \int_{\text{Box}(0, \frac{\varkappa}{4})} |\langle \xi, z \rangle|^2 \leq \frac{n\varkappa^\nu}{2^{\nu+1}} \varepsilon^2 |\xi|^2. \end{aligned}$$

Hence, $|A - I| \leq \sqrt{\frac{n}{2}} \left(\frac{\varkappa}{2}\right)^{n+1} \varepsilon$ and A is a nondegenerate complex $n \times n$ matrix if $\varepsilon < \sqrt{\frac{2}{n}} \left(\frac{2}{\varkappa}\right)^{n+1}$.

For the positive definite Hermitian $n \times n$ matrix A^*A , there exists a unitary matrix $U \in U(n)$ such that UA^*AU^* is a real diagonal matrix $\text{diag}\{\mu_1, \dots, \mu_n\}$, $\mu_i > 0$ (for instance, see [19]). Hence, there are two orthonormal bases $\{w_i = U^*e_i\}_{i=1, \dots, n}$ and $\{v_i = \frac{1}{\lambda_i}Aw_i\}_{i=1, \dots, n}$, where $\lambda_i = \sqrt{\mu_i} > 0$ and $Aw_i = \lambda_i v_i$ for $i = 1, \dots, n$. Here $\{e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)\}_{i=1, \dots, n}$ is the standard basis of \mathbb{C}^n .

Consider the unitary matrix $V \in U(n)$ with $Vv_i = w_i$ for $i = 1, \dots, n$. Since $VAw_i = V(\lambda_i v_i) = \lambda_i w_i$, the matrix VA is diagonal in the basis $\{w_1, \dots, w_n\}$, and hence, Hermitian in the origin basis $\{e_i\}_{i=1, \dots, n}$. Therefore, $A(Vu) = VA(u)$ is a Hermitian matrix, and consequently, $K(Vu) = 0$. Thus, we have demonstrated that $P(Vu) = a(Vu) \equiv \text{const}$.

Estimate $|V - I|$. Since $|Aw_i - w_i| = |\lambda_i v_i - w_i| \leq \sqrt{\frac{n}{2}} \left(\frac{\varkappa}{2}\right)^{n+1} \varepsilon$ for all $i = 1, \dots, n$, we obtain

$$|\lambda_i - 1| = ||Aw_i| - |w_i|| \leq |Aw_i - w_i| \leq \sqrt{\frac{n}{2}} \left(\frac{\varkappa}{2}\right)^{n+1} \varepsilon$$

and

$$|v_i - w_i| \leq |\lambda v_i - w_i| + |\lambda_i v_i - v_i| \leq \sqrt{2n} \left(\frac{\varkappa}{2}\right)^{n+1} \varepsilon.$$

Given a vector $\xi = \sum_{i=1}^n \xi_i v_i \in \mathbb{C}^n$, we have

$$|(V - I)\xi| = \left| \sum_{i=1}^n \xi_i w_i - \xi_i v_i \right| \leq |\xi| \sqrt{\sum_{i=1}^n |w_i - v_i|^2} \leq \sqrt{2} |\xi|^2 n^2 \left(\frac{\varkappa}{2}\right)^{n+1} \varepsilon.$$

Hence, $|V - I| < \frac{n\varkappa^{n+1}}{2^n} \varepsilon$. □

4 Local Geometric Rigidity

4.1 Qualitative local rigidity

Lemma 6. *For every $q \in (0, 1)$, there exist nondecreasing functions $\mu_i(\cdot, q): [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2$, such that*

$$(1) \quad \mu_i(t, q) \rightarrow 0 \text{ as } t \rightarrow 0, \quad i = 1, 2;$$

(2) for each mapping f of class $I(1 + \varepsilon, B(0, 1))$, where $B(0, 1) \subset \mathbb{H}^n$, there exists an isometry θ satisfying

$$\rho(f(x), \theta(x)) \leq q \mu_1(\varepsilon, q) \quad \text{for all } x \in B(0, q),$$

$$\|D_h f(x) - D_h \theta(x)\|_{2, B(0, q)} \leq |B(0, q)|^{1/2} \mu_2(\varepsilon, q).$$

Proof. Put $B = B(0, 1)$,

$$\mu_1(\varepsilon, q) = \frac{1}{q} \sup_{f \in I(1+\varepsilon, B)} \inf \left\{ \sup_{x \in B(0, q)} \rho(f(x), \varphi(x)) : \varphi \text{ is an isometry} \right\}$$

and

$$\mu_2(\varepsilon, q) = \sup_{f \in I(1+\varepsilon, B)} \inf \left\{ \frac{\|D_h f - D_h \varphi\|_{2, B(0, q)}}{|B(0, q)|^{1/2}} : \varphi \text{ is an isometry} \right\}.$$

Property (2) is obvious.

It remains to prove that μ_1 and μ_2 enjoy property (1).

(I) Assume that for some $q \in (0, 1)$ the function $\mu_1(t, q)$ fails to tend to 0 as $t \rightarrow 0$. Then there exist $\delta > 0$ and a sequence of quasi-isometries $\{f_j \in I(L_j, B)\}$ with $L_j < 1 + \frac{1}{j}$ such that

$$\sup_{x \in B(0, q)} \rho(f_j(x), \varphi(x)) \geq \varepsilon \quad \text{for all } j \in \mathbb{N} \quad (10)$$

for every isometry φ . Since the isometry group contains translations and reflections, we may assume that $f_j(0) = 0$ and $J(x, f) > 0$ almost everywhere on $B(0, 1)$. By Lemma 1, the sequence $\{f_j\}$ is an equicontinuous and uniformly bounded family on every domain compactly embedded into $B(0, 1)$, for example, on the ball $B(0, q)$. Consequently, there exists a mapping $f_0: B(0, q) \rightarrow \mathbb{H}^n$ and a subsequence uniformly converging to f_0 , which we also denote by $\{f_j\}$. Since all quasi-isometric mappings are of bounded distortion, by [26] f_0 is a mapping with 1-bounded distortion. Verify that f_0 is an isometry.

The weak convergence of the Jacobians [26] yields

$$\lim_{j \rightarrow \infty} \int_B J(x, f_j) \xi(x) dx = \int_B J(x, f_0) \xi(x) dx$$

for every $\xi \in C_O(B)$. On the other hand,

$$L_j^{-\nu} \int_B \xi(x) dx \leq \int_B J(x, f_j) \xi(x) dx \leq L_j^\nu \int_B \xi(x) dx.$$

Consequently, $J(x, f_0) \equiv 1$ almost everywhere on $B(0, q)$. This is possible only if f_0 is an isometry. Applying (10) for f_0 , we arrive at a contradiction.

(II) Now we prove property (1) for μ_2 . Assume the contrary. Then there exist numbers $\varepsilon > 0$, $q \in (0, 1)$, a sequence of quasi-isometric mappings $\{f_j \in I(L_j, B)\}$ with $L_j < 1 + \frac{1}{j}$, and

a sequence of isometries θ_j such that

$$\sup_{x \in B(0,q)} \rho(f_j(x), \theta_j(x)) \leq \mu_1(L_j - 1, q) \quad \text{and} \quad \|D_h f_j(x) - D_h \theta_j(x)\|_{2,B(0,q)} dx \geq \varepsilon.$$

Like in part (I) of the proof, we may assume that the sequence $\{f_j\}$ converges to an isometry f_0 uniformly on the ball $B(0, q)$. Clearly, the mappings θ_j converge to f_0 uniformly on $B(0, q)$ as $j \rightarrow \infty$. Therefore, $|D_h \theta_j(x) - D_h f_0(x)| \rightarrow 0$ and $|D_h f_j(x)| \rightarrow 1 = |D_h f_0(x)|$ as $j \rightarrow \infty$ for all $x \in B(0, q)$. Since the space W_2^1 is uniformly convex, the convergence of the norms along with the uniform convergence $f_j \rightarrow f_0$ imply the convergence $\int_{B(0,q)} |D_h f_j(x) - D_h f_0(x)|^2 dx \rightarrow 0$. The properties of uniformly convex spaces can be found in [5]. We arrive at a contradiction:

$$\begin{aligned} \varepsilon &\leq \|D_h f_j(x) - D_h \theta_j(x)\|_{2,B(0,q)} \\ &\leq \|D_h f_j(x) - D_h f_0(x)\|_{2,B(0,q)} + \|D_h \theta_j(x) - D_h f_0(x)\|_{2,B(0,q)} \xrightarrow{j \rightarrow \infty} 0. \end{aligned}$$

□

4.2 Application of the operator Q

In this section we apply the coercive estimate for the operator Q . In view of the connection between the Lie algebra of isometries and the kernel of Q , we obtain the following lemma, which shows that we can slightly perturb the isometry of Lemma 6 to make the projection vanish.

Lemma 7. *Take $n > 1$. There exist constants $c_1 = c_1(n) > 0$ and $\varepsilon_1 = \varepsilon_1(n) > 0$ and a nondecreasing function $\mu_3: [0, \varepsilon_1) \rightarrow [0, \infty)$ with $\mu_3(t) \rightarrow 0$ as $t \rightarrow 0$ such that, given a ball $B(a, r) \subset \mathbb{H}^n$ and a mapping $f \in I(1 + \varepsilon, B(a, r))$ with $\varepsilon < \varepsilon_1$, there is an isometry θ satisfying*

$$\begin{aligned} \|D_h f - D_h \theta\|_{2,B(a, \frac{3r}{10})} &\leq c_1 \|Q(x^{-1} \cdot (\theta^{-1} \circ f)(x))\|_{2,B(a, \frac{3r}{10})}, \\ \|D_h f - D_h \theta\|_{2,B(a, \frac{r}{2})} &\leq \left| B\left(a, \frac{r}{2}\right) \right|^{1/2} \mu_3(\varepsilon). \end{aligned}$$

Here Q is the differential operator (8).

Proof. Assume first that $B(a, r) = B(0, 1)$. Consider a mapping $f \in I(1 + \varepsilon, B(0, 1))$. By Lemma 6, there exists an isometry φ such that the mapping $g = \varphi^{-1} \circ f \in I(1 + \varepsilon, B(0, 1))$ satisfies

$$|\tilde{g}(x) - z| \leq \rho(g(x), x) = \rho(f(x), \varphi(x)) \leq \frac{\mu_1(\varepsilon, 1/2)}{2}$$

for all $x = (z, t) \in B(0, 1/2)$ and

$$\int_{B(0, 1/2)} |D_h g(x) - I|^2 dx = \int_{B(0, 1/2)} |D_h f(x) - D_h \varphi(x)|^2 dx \leq (\mu_2(\varepsilon, 1/2))^2 |B(0, 1/2)|.$$

(Here \widetilde{g} stands for the projection of g onto the first n complex coordinates.)

Take $\varepsilon < \varepsilon_1$, where $\frac{\mu_1(\varepsilon_1, 1/2)}{2} \leq \sqrt{\frac{2}{n}} \left(\frac{2}{\varkappa}\right)^{n+1}$. By Lemma 5, there exists a matrix $V \in U(n)$ such that $|V - I| < \frac{n\varkappa^{n+1}}{2^{n+1}} \mu_1(\varepsilon, 1/2)$ and $D_h P(V\widetilde{g}) \equiv 0$.

Put $\theta^{-1} = \varphi_V \circ \varphi^{-1}$. We have $D_h P(x^{-1} \cdot (\widetilde{\theta^{-1} \circ f})(x)) = D_h P(\widetilde{x^{-1}}) + D_h P(V\widetilde{g}) \equiv 0$. By the coercive estimate [13], there is a constant $c_1 = c_1(n) > 0$ such that

$$\|D_h f - D_h \theta\|_{2, B(0, 3/10)} = \|D_h(\theta^{-1} \circ f) - I\|_{2, B(0, 3/10)} \leq c_1 \|Q(x^{-1} \cdot (\theta^{-1} \circ f)(x))\|_{2, B(0, 3/10)}.$$

We have

$$|D_h f - D_h \theta| = |V D_h g - I| \leq |D_h g - I| + |V - I|.$$

Hence,

$$\frac{\|D_h f - D_h \theta\|_{2, B(0, 1/2)}}{|B(0, 1/2)|^{1/2}} \leq \mu_2(\varepsilon, 1/2) + \frac{n\varkappa^{n+1}}{2^{n+1}} \mu_1(\varepsilon, 1/2) = \mu_3(\varepsilon).$$

To complete the proof, consider an arbitrary ball $B(a, r)$ and a mapping f of class $I(1 + \varepsilon, B(a, r))$. Then the mapping $g = \delta_{\frac{1}{r}} \circ \pi_{-a} \circ f \circ \pi_a \circ \delta_r$ belongs to the class $I(1 + \varepsilon, B(0, 1))$. Hence, there is an isometry ψ close to g satisfying the estimates of the lemma. Then $\theta = \pi_a \circ \delta_r \circ \psi \circ \delta_{\frac{1}{r}} \circ \pi_{-a}$ is a required isometry for f . \square

4.3 Quantitative local rigidity

Lemma 8. *Tale $n > 1$. Given a ball $B(a, r) \subset \mathbb{H}^n$ and a mapping $f \in I(1 + \varepsilon, B(a, r))$, there is an isometry φ satisfying*

$$\frac{1}{|B(a, \frac{3r}{10})|} \int_{B(a, \frac{3r}{10})} |D_h f(x) - D_h \varphi(x)|^2 dx \leq (c_2 \varepsilon)^2.$$

The constant c_2 depends only on n .

Proof. Put $B_1 = B(a, \frac{3r}{10})$, $B_2 = B(a, \frac{r}{3})$, $B_3 = B(a, \frac{r}{2})$. By Lemma 7, there is an isometry θ such that

$$\begin{aligned} \|D_h(\theta^{-1} \circ f) - I\|_{2, B_1} &\leq c_1 \|Q(x^{-1} \cdot (\theta^{-1} \circ f)(x))\|_{2, B_1}, \\ \|D_h(\theta^{-1} \circ f) - I\|_{2, B_3} &\leq |B_3|^{1/2} \mu_3(\varepsilon). \end{aligned}$$

Put $g = \theta^{-1} \circ f \in I(1 + \varepsilon, B(a, r))$. By Proposition 4 of [12], there is a number $\varepsilon_2 > 0$ such that g preserves KR -orientation on B_1 if $\varepsilon < \varepsilon_2$. Thus, assuming that $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, we may apply Lemmas 3 and 7. We obtain

$$\|D_h g - I\|_{2, B_1} \leq c_1 \frac{\varepsilon^2 + 2\varepsilon}{2} (\|D_h g - I\|_{2, B_1} + 2|B_1|^{1/2}) + \frac{c_1}{2} \|D_h g - I\|_{4, B_1}^2.$$

Estimate $\int_{B_1} |D_h g(x) - I|^4 dx$. Consider an arbitrary ball $B = B(a_0, r_0) \subset B_2$. Then $B(a_0, 2r_0) \subset B(a, r)$ and, by Lemma 7, there is an isometry θ_B , and hence a matrix $A_B = D_h \theta_B \in U(n)$ such that

$$\int_B |D_h g(x) - A_B|^2 dx \leq (\mu_3(\varepsilon))^2 |B| = (\mu_3(\varepsilon))^2 \int_B |A_B|^2 dx.$$

For the ball B_2 , we have $A_{B_2} = I$ and

$$\int_{B_2} |D_h g(x) - I|^2 dx \leq \int_{B_3} |D_h g(x) - I|^2 dx \leq |B_2| (3/2)^\nu (\mu_3(\varepsilon))^2.$$

It follows that $D_h g$ is a mapping with bounded specific oscillation in the sense of L_2 relative to the class $U(n)$ on the ball B_2 (see [11, Definitions 1 and 2]). Thus, by the Corollary to Theorem 1 of [11], there are constants $C, \sigma_0 > 0$ such that on the ball $B_1 = \frac{9}{10} B_2$ we have

$$\int_{B_1} |D_h g(x) - I|^4 dx \leq C (3/2)^\nu (\mu_3(\varepsilon))^2 \int_{B_1} |D_h g(x) - I|^2 dx$$

provided that $(3/2)^{n+1} \mu_3(\varepsilon) < \frac{\sigma_0}{4}$. We need to consider $\varepsilon < \varepsilon_3$, where $\varepsilon_3 \leq \min\{\varepsilon_1, \varepsilon_2\}$ and $(3/2)^{n+1} \mu_3(\varepsilon_3) \leq \frac{\sigma_0}{4}$.

Finally,

$$\|D_h g - I\|_{2, B_1} \leq \frac{c_1 \varepsilon (\varepsilon + 2)}{2} (\|D_h g - I\|_{2, B_1} + 2|B_1|^{1/2}) + \frac{c_1 \sqrt{C} 3^{n+1} \mu_3(\varepsilon)}{2^{n+2}} \|D_h g - I\|_{2, B_1}.$$

Take $\varepsilon_4 \leq \varepsilon_3$ such that

$$\frac{c_1 \sqrt{C} 3^{n+1} \mu_3(\varepsilon_4)}{2^{n+2}} \leq \frac{1}{4}, \quad \frac{c_1 \varepsilon_4 (\varepsilon_4 + 2)}{2} \leq \frac{1}{4}.$$

If $\varepsilon < \varepsilon_4$ then

$$\|D_h g - I\|_{2, B_1} = \|D_h f - D_h \theta\|_{2, B_1} \leq 2c_1 (\varepsilon_4 + 2) \varepsilon |B_1|^{1/2}.$$

Thus, we have established the lemma for $f \in I(1 + \varepsilon, B(a, r))$ with $\varepsilon < \varepsilon_4$. In the case $\varepsilon \geq \varepsilon_4$, given an isometry φ , we obviously have

$$\frac{1}{|B_1|} \int_{B_1} |D_h f(x) - D_h \varphi(x)|^2 dx \leq (2 + \varepsilon)^2 \leq \left(\frac{2}{\varepsilon_4} + 1\right)^2 \varepsilon^2.$$

The lemma is proved with the constant $c_2 = \max\{2c_1(\varepsilon_4 + 2), \frac{2}{\varepsilon_4} + 1\}$. \square

5 Global Geometric Rigidity

In this section we prove Theorem 1. Local rigidity (Lemma 8) means, in particular, that the horizontal differential of a quasi-isometry is a BMO mapping. To pass from local rigidity to

global rigidity, we apply the John–Nirenberg technique. In the Euclidean case, a necessary and sufficient condition for the exponential integrability of a BMO mapping is that U is a Hölder domain [22, 10]. In the metric space setting, this also holds (see [3] for instance). Thus, we can prove global geometric rigidity in the Sobolev norm on Hölder domains. Note that we can prove geometric rigidity in the uniform norm only on John domains.

To begin with, we give definitions of John and Hölder domains and some of their properties on a metric space (\mathbb{X}, ρ) . For a domain $U \subset \mathbb{X}$, denote the distance from a point $x \in U$ to the boundary ∂U by $\rho_U(x) = \text{dist}(x, \partial U)$. For a ball $B \subset \mathbb{X}$, denote by $x(B)$ and $r(B)$ its center and radius respectively.

Definition 6 ([14, 2]). A bounded open proper subset U of a metric space (\mathbb{X}, ρ) with a distinguished point $x_* \in U$ is called a (*metric*) *John domain* if it satisfies the following “twisted cone” condition: there exist constants $\beta \geq \alpha > 0$ such that for all $x \in U$ there is a curve $\gamma : [0, l] \rightarrow U$ with $l \leq \beta$ parameterized by arc length such that $\gamma(0) = x$, $\gamma(l) = x_*$ and $\rho_U(\gamma(s)) \geq \frac{\alpha}{l}s$.

The numbers α and β are the *inner* and *outer radii* of U .

Definition 7. A proper open subset U of a metric space (\mathbb{X}, ρ) is a *Hölder domain* if there exists a constant $H > 0$ such that for every $x \in U$ we can find a path γ joining x with the distinguished point $x_* \in U$ satisfying

$$\int_{\gamma} \frac{ds}{\rho_U(\gamma(s))} \leq H \ln \left(\frac{H}{\rho_U(x)} \right),$$

where ds is the arc-length measure.

The reader may recognize the above integral as the quasihyperbolic length of γ . Hölder domains are also known as domains satisfying a quasihyperbolic boundary condition.

It is easy to verify that every John domain is a Hölder domain. Note that the notions of John and Hölder domains are independent of the choice of the equivalent metrics.

Theorem 1 is a particular case of the following

Theorem 2. Consider a Hölder domain U on the Heisenberg group \mathbb{H}^n , $n > 1$. For every $f \in I(1 + \varepsilon, U)$ there exists an isometry θ satisfying

$$\int_U \exp \left(\frac{N_1 |D_h f(x) - D_h \theta(x)|}{\varepsilon} \right) dx \leq 16|U|.$$

The constant N_1 depends on n , H , and $\rho_U(x_*)/\text{diam}(U)$.

Lemma 9. Suppose U is a Hölder or John domain in a metric space (\mathbb{X}, ρ) . Then, for every point $x \in U$ there is a chain of balls $B_i = B(x_i, r_i)$, $i = 0, \dots, k$, with $B_0 = B(x_*, \frac{\rho_U(x_*)}{4})$, satisfying the following conditions:

- (1) $x_0 = x_*$ and $x_k = x$;
- (2) if $0 \leq i < k$ then $\frac{7}{9}r_{i+1} \leq r_i \leq \frac{9}{7}r_{i+1}$ and there is a ball $G_i \subset B_i \cap B_{i+1}$ with $r(G_i) = \frac{1}{2} \min\{r_i, r_{i+1}\}$;
- (3) $4B_i \subset U$ for all $i = 0, \dots, k$;
- (4) $k < 9H \ln \frac{H}{4r_k}$ if U is a Hölder domain and $k < 9\frac{\beta}{\alpha} \ln \frac{8\beta}{r_k}$ if U is a John domain;
- (5) $B_k \subset (1 + 5\frac{\beta}{\alpha})B_i$ and $B_k \subset (3 + 10\frac{\beta}{\alpha})G_i$ for all $i = 0, \dots, k-1$ if U is a John domain.

Proof. Fix a point $x \in U$. Construct a chain B_0, B_1, \dots, B_k of balls $B_i = B(x_i, r_i)$ with $r_i = \frac{\text{dist}(x_i, \partial U)}{4}$ for all $i = 0, \dots, k$ and $x_0 = x_*$, $x_k = x$.

Thus, we must find the number k and the points x_1, \dots, x_{k-1} . Consider a rectifiable curve γ joining x with x_* and satisfying the conditions of Definition 6 or Definition 7. Parameterize γ by arc length. Put $s_0 = l$ and $x_0 = x_* = \gamma(l) = \gamma(s_0)$. Assume by induction that x_0, \dots, x_i are known and put $x_{i+1} = \gamma(s_{i+1})$, where $s_{i+1} = \inf\{s : \gamma|_{[s, s_i]} \subset B(x_i, \frac{r_i}{2})\}$. The process stops on step j if the ball $\frac{1}{2}B_j$ intersects the ball $B(x, \frac{\text{dist}(x, \partial U)}{8})$; then, put $j = k-1$.

Conditions (1) and (3) obviously hold.

(2) By construction, $4r_i = \text{dist}(x_i, \partial U)$ for $i = 1, \dots, k$, and $\rho(x_i, x_{i+1}) \leq \frac{r_i}{2} + \frac{r_{i+1}}{2}$ for $i = 0, \dots, k-1$. Hence, $4r_i \leq 4r_{i+1} + \frac{r_i}{2} + \frac{r_{i+1}}{2}$ and $4r_{i+1} \leq 4r_i + \frac{r_i}{2} + \frac{r_{i+1}}{2}$. Therefore, $\frac{7}{9}r_{i+1} \leq r_i \leq \frac{9}{7}r_{i+1}$. Since $\frac{1}{2}B_i \cap \frac{1}{2}B_{i+1} \neq \emptyset$ for $i = 0, \dots, k-1$, there is a ball G_i such that $G_i \subset B_i \cap B_{i+1}$, $x(G_i) \in \frac{1}{2}B_i \cap \frac{1}{2}B_{i+1}$, and $r(G_i) = \frac{1}{2} \min\{r_i, r_{i+1}\}$.

(4) By construction, $\frac{r_i}{2} = \rho(\gamma(s_i), \gamma(s_{i+1})) \leq l(\gamma|_{[s_{i+1}, s_i]})$ for all $i = 0, \dots, k-2$ and $\frac{r_k}{2} < \rho(\gamma(s_{k-1}), \gamma(0)) \leq l(\gamma|_{[0, s_{k-1}]})$. Hence, $\sum_{i=0}^{k-2} r_i + r_k \leq 2l$. If $y \in \frac{1}{2}\overline{B}_i$ then $\rho_U(y) \leq \rho_U(x_i) + \rho(x_i, y) \leq \frac{9}{2}r_i$ for all $i = 0, \dots, k$. Thus,

$$\int_{\gamma([s_{i+1}, s_i])} \frac{ds}{\rho_U(\gamma(s))} \geq \int_{\gamma([s_{i+1}, s_i])} \frac{2ds}{9r_i} \geq \frac{1}{9} \quad \text{for all } i = 0, \dots, k-2$$

and

$$\int_{\gamma([0, s_{k-1}])} \frac{ds}{\rho_U(\gamma(s))} \geq \int_{\gamma([0, s_{k-1}]) \cap \frac{1}{2}B_k} \frac{2ds}{9r_k} \geq \frac{1}{9}.$$

If U is a Hölder domain then $\frac{k}{9} \leq \int_{\gamma} \frac{ds}{\rho_U(\gamma(s))} \leq H \ln \frac{H}{4r_k}$. If U is a John domain then $\frac{r_k}{2} \leq s' \leq \frac{l}{\alpha} \rho_U(\gamma(s')) \leq \frac{9l}{2\alpha} r_k$ for $s' = \sup\{s : \gamma|_{[0, s]} \subset \frac{1}{2}B_k\} < s_{k-1}$. We have

$$\begin{aligned} \frac{k}{9} &\leq \int_{\gamma} \frac{ds}{\rho_U(\gamma(s))} = \int_{s'}^l \frac{ds}{\rho_U(\gamma(s))} + \int_0^{s'} \frac{ds}{\rho_U(\gamma(s))} \leq \int_{s'}^l \frac{lds}{\alpha s} + \int_0^{s'} \frac{2ds}{7r_k} \\ &= \frac{l}{\alpha} (\ln l - \ln s') + \frac{2}{7r_k} s' \leq \frac{l}{\alpha} \left(\ln l - \ln \left(\frac{r_k}{2} \right) + \frac{9}{7} \right) \leq \frac{\beta}{\alpha} \ln \frac{8\beta}{r_k}. \end{aligned}$$

(5) Assume that U is a John domain. For $i \in \{0, \dots, k-1\}$ we have $\rho(x, x_i) \leq l(\gamma|_{[0, s_i]}) = s_i \leq \frac{\beta}{\alpha} \rho_U(x_i) = 4\frac{\beta}{\alpha} r_i$ and

$$4r_k = \rho_U(x) \leq \rho(x, x_i) + \rho_U(x_i) \leq \left(\frac{4\beta}{\alpha} + 4 \right) r_i.$$

This yields $\rho(x_i, z) \leq \rho(x, x_i) + r_k \leq (1 + \frac{5\beta}{\alpha})r_i$ for every $z \in B_k$. Thus, $B_k \subset (1 + 5\frac{\beta}{\alpha})B_i$.

Suppose $r(G_i) = \frac{r_i}{2}$, where j equals either i or $i + 1$. Then

$$\rho(x(G_i), z) \leq \rho(x(G_i), x_j) + \rho(x_j, z) \leq \left(1 + 2\left(1 + 5\frac{\beta}{\alpha}\right)\right)\frac{r_j}{2}$$

for all $z \in (1 + 5\frac{\beta}{\alpha})B_j$. Hence, $B_k \subset (3 + 10\frac{\beta}{\alpha})G_i$. \square

Below we need the following result asserting that the boundary of a Hölder domain or John domain is regular in some sense.

Lemma 10. *Given a Hölder domain U in \mathbb{H}^n , there is a constant $0 < \tau < 1$ depending only on n , H , and $\frac{\rho_U(x_*)}{\text{diam } U}$ such that $\int_U \frac{dx}{\rho_U(x)^\tau} \leq \frac{2|U|}{\rho_U(x_*)^\tau}$.*

Given a John domain U , there is a constant $0 < \tau_0 < 1$ depending only on n such that $\int_U \frac{dx}{\rho_U(x)^\tau} \leq \frac{2|U|}{\alpha^\tau}$ with $\tau = \tau_0(\frac{\alpha}{\beta})^\nu$.

Proof. The first part (on Hölder domains) is Theorem 3.3 of [3]. We estimate τ in the case of John domains.

Consider a countable family of balls \mathcal{D} covering U such that $\{\frac{1}{5}D\}_{D \in \mathcal{D}}$ is a disjoint family, $\sum_{D \in \mathcal{D}} \chi_D(x) \leq N$ for all $x \in U$, and $r(D) = \frac{1}{4}\rho_U(x(D))$ for every $D \in \mathcal{D}$.

Fix a ball $D \in \mathcal{D}$. By Lemma 9, there is a chain of balls $B_0, \dots, B_k = D$ that covers the curve γ joining $x(D)$ and x_* . Since the family \mathcal{D} covers U there is a chain of balls $D_0, \dots, D_l = D$ in \mathcal{D} covering γ and satisfying $x_* \in D_0$ and $D_i \cap D_{i+1} \neq \emptyset$ for $i = 0, \dots, l - 1$. We can take the same D_0 for all $D \in \mathcal{D}$.

Put $D_i = B(y_i, \rho_i)$. Since $D_i \cap D_{i+1} \neq \emptyset$, it follows that $\frac{3}{5}\rho_{i+1} \leq \rho_i \leq \frac{5}{3}\rho_{i+1}$ for $i = 0, \dots, l - 1$. Suppose that $\rho_l \leq \rho_0$. Then $\rho_0 \leq (\frac{5}{3})^l \rho_l$ and hence $l \geq \log_{\frac{5}{3}} \frac{\rho_0}{\rho_l}$.

The chains $\{B_j\}$ and $\{D_i\}$ cover γ . Hence, each D_i intersects some ball B_j . It follows that $\frac{3}{5}r_j \leq \rho_i \leq \frac{5}{3}r_j$. Therefore, for $y \in D$ we have

$$\rho(y, y_i) \leq \rho(y, x_j) + \rho(x_j, y_i) \leq (1 + 5\frac{\beta}{\alpha})r_j + r_j + \rho_i \leq \frac{\rho_i}{3}(13 + 25\frac{\beta}{\alpha}) < 13\frac{\beta}{\alpha}\rho_i.$$

Thus, $D \subset hD_i$ for all $i = 0, \dots, l$ with $h = 13\frac{\beta}{\alpha}$. Putting $S(x) = \sum_{D \in \mathcal{D}} \chi_{hD}(x)$, we obtain $S(x) \geq \log_{\frac{5}{3}} \frac{\rho_0}{r(D)}$ if $x \in D$ and $r(D) \leq \rho_0$.

It is known (see [3, Lemma 3.4] for example), that there exists a constant $C > 1$ depending only on n such that $\|S\|_{p, \mathbb{H}^n} \leq Cph^\nu \|\sum_{D \in \mathcal{D}} \chi_{\frac{1}{5}D}\|_{p, \mathbb{H}^n} \leq Cph^\nu |U|^{1/p}$ for all $p \geq 1$. Hence

$$\|e^{aS}\|_{1,U} \leq |U| + \sum_{m>1} \frac{a^m \|S\|_{m,U}^m}{m!} \leq |U| \left(1 + \sum_{m>1} \frac{(Ch^\nu ma)^m}{m!}\right) \leq 2|U| \quad \text{if } a = \frac{1}{2Ch^\nu e}.$$

For $b = a(\ln \frac{5}{3})^{-1} > 0$, we have

$$\begin{aligned} \int_U \frac{dx}{\rho_U(x)^b} &\leq \sum_{D \in \mathcal{D}} \frac{|D|}{3^b r(D)^b} \leq \sum_{D \in \mathcal{D}, r(D) \leq \rho_0} \frac{5^\nu}{3^b \rho_0^b} \int_{\frac{1}{5}D} e^{bS(x) \ln \frac{5}{3}} dx + \sum_{D \in \mathcal{D}, r(D) > \rho_0} \frac{|D|}{3^b \rho_0^b} \\ &\leq \frac{5^\nu}{3^b \rho_0^b} \int_U e^{bS(x) \ln \frac{5}{3}} dx + \frac{5^\nu}{3^b \rho_0^b} |U| \leq \frac{3 \cdot 5^\nu 5^b}{3^b \alpha^b} |U| \leq 5^{\nu+1} |U| \frac{1}{\alpha^b}. \end{aligned}$$

Here we have used the fact that $\alpha \leq \rho_U(x_*) \leq \rho(y_0, x_*) + \rho_U(y_0) \leq 5\rho_0$ and the inequality $b \leq 1$. Applying Hölder's inequality, we obtain the desired inequality for $\tau = \frac{b}{3^\nu} = \tau_0(\frac{\alpha}{\beta})^\nu$ where $\tau_0 = \frac{1}{6C\nu 13^\nu e \ln \frac{5}{3}}$. \square

Proof of Theorem 2. Consider a mapping $f \in I(1 + \varepsilon, U)$, where U is a Hölder domain. Put $F(x) = D_h f(x)$. By Lemma 8 for every ball B with $\frac{10}{3}B \subset U$ there is a unitary matrix A_B such that

$$\int_B |F(x) - A_B| dx \leq |B|^{1/2} \left(\int_B |F(x) - A_B|^2 dx \right)^{1/2} \leq \sigma |B|$$

with $\sigma = c_2 \varepsilon$. Thus, it is easy to see that $D_h f$ is a *BMO* mapping. By [3, Theorem 2.2],

$$\int_{B'} \exp(C_1 \sigma^{-1} |F(x) - F_{B'}|) dx \leq 16 |B'|,$$

where $B' = \frac{1}{2}B$, $C_1 = \frac{1}{12}$, and $F_{B'}$ is the mean value of F over the ball B' . The proof of this fact goes along the same lines as the proof of the classical John–Nirenberg Theorem. Consequently,

$$\int_{B'} e^{C_1 \sigma^{-1} |F(x) - A_B|} dx \leq e^{C_1 \sigma^{-1} |F_{B'} - A_B|} \int_{B'} e^{C_1 \sigma^{-1} |F(x) - F_{B'}|} dx \leq 16 e^{2^\nu C_1} |B'|.$$

Consider the family of balls $\{B(x, \frac{\text{dist}(x, \partial U)}{8})\}_{x \in U}$. We can choose a countable subfamily \mathcal{F} such that $\bigcup_{B \in \mathcal{F}} B = U$ and $\{\frac{1}{5}B \mid B \in \mathcal{F}\}$ is a disjoint family.

Put $A_* = A_{B_0}$, where $B_0 = B(x_*, \frac{\text{dist}(x_*, \partial U)}{4})$. For every $B \in \mathcal{F}$, there is a chain of balls $B_0, \dots, B_k = 2B$ satisfying conditions (1)–(4) of Lemma 9. Obviously,

$$|A_{B_i} - A_{B_{i+1}}| \leq |G_i|^{-1} \int_{B_i} |F(x) - A_{B_i}| dx + |G_i|^{-1} \int_{B_{i+1}} |F(x) - A_{B_{i+1}}| dx \leq \sigma C_2$$

with $C_2 = 2^\nu (1 + (\frac{9}{7})^\nu)$. For $0 < C_3 < C_1$, it follows that

$$\int_B e^{C_3 \sigma^{-1} |F(x) - A_*|} dx \leq \prod_{i=0}^{k-1} e^{C_3 \sigma^{-1} |A_{B_i} - A_{B_{i+1}}|} \int_B e^{C_3 \sigma^{-1} |F(x) - A_{B_k}|} dx \leq 16 e^{k C_2 C_3} e^{2^\nu C_1} |B|$$

Applying $k \leq 9H \ln \frac{H}{8r(B)}$, we obtain

$$\int_U e^{C_3 \sigma^{-1} |F(x) - A_*|} dx \leq \sum_{B \in \mathcal{F}} \int_B e^{C_3 \sigma^{-1} |F(x) - A_*|} dx \leq C_4 \int_U \frac{dx}{\rho_U(x)^{9HC_2C_3}} < \frac{2C_4}{\rho_U(x_*)^\tau} |U| \quad (11)$$

if C_3 is small enough so that $9HC_2C_3 \leq \tau$, where τ is as in Lemma 10. \square

Proof of Theorem 1. Consider a John domain U with a distinguished point x_* and a mapping f of class $I(1 + \varepsilon, U)$. Put $B_* = B(x_*, r_*)$, where $r_* = \frac{\text{dist}(x_*, \partial U)}{4}$.

The proof of the first assertion follows verbatim the proof of Theorem 2 till relation (11).

We rearrange (11) as

$$\int_U e^{C_3 \sigma^{-1} |D_h f(x) - A_*|} dx \leq C_4 \beta^{9 \frac{\beta}{\alpha} C_2 C_3} \int_U \frac{dx}{\rho_U(x)^{9 \frac{\beta}{\alpha} C_2 C_3}} \leq \frac{2C_4 \beta^\tau |U|}{\alpha^\tau} \leq 4C_4 |U|$$

if $9 \frac{\beta}{\alpha} C_2 C_3 < \tau = \tau_0 \left(\frac{\alpha}{\beta}\right)^\nu$ where $C_4 = 64 \cdot 2^{2^\nu C_1} 5^\nu$. Here we use the fact that $\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\beta}} < 2$ and, consequently, $\left(\frac{\beta}{\alpha}\right)^\tau < 2$ since $\tau_0 < 1$. By Hölder's inequality, we obtain the desired inequality with $N_1 = \frac{C_3}{2+\nu+2^{\nu-3}} = C' \left(\frac{\alpha}{\beta}\right)^{\nu+1}$.

Let us now prove the second assertion.

Consider a point $x \in U$ and the chain B_0, \dots, B_k of Lemma 9. Since balls are John domains and, consequently, Hölder domains, Theorem 2 implies that, for each $i = 0, \dots, k$, there is an isometry θ_i such that

$$\int_{4B_i} \exp \left\{ \frac{N_1 |D_h f(x) - D_h \theta_i(x)|}{\varepsilon} \right\} dx \leq 16 |4B_i|.$$

Hence, $\|D_h f - D_h \theta_i\|_{\nu+1, 4B_i} \leq \left(\frac{\varepsilon}{N_1}\right) (16 |4B_i|)^{1/(\nu+1)}$, and, by Lemma 11, we conclude that $\rho(f(y), \theta_i(y)) \leq \omega r_i$ for all $y \in B_i$ with $\omega = C_1(\sqrt{\varepsilon} + \varepsilon)$.

We have

$$\rho(f(x), \theta_0(x)) \leq \rho(f(x), \theta_k(x)) + \sum_{i=0}^{k-1} \rho(\theta_i(x), \theta_{i+1}(x))$$

and

$$\rho(\theta_i(y), \theta_{i+1}(y)) \leq \rho(f(y), \theta_i(y)) + \rho(f(y), \theta_{i+1}(y)) \leq \omega(r_i + r_{i+1}) \leq \frac{32}{7} \omega r(G_i)$$

for every $y \in G_i$. Consider the case $\frac{32}{7} \omega < \frac{1}{2}$. Lemma 12 yields

$$\rho(\theta_i(y), \theta_{i-1}(y)) \leq C_2 \omega r(G_i),$$

where $C_2 = \frac{160}{7} (3 + 10 \frac{\beta}{\alpha})$, for all $y \in B(x, r) = B_k \subset (3 + 10 \frac{\beta}{\alpha}) G_i$ and all $i = 1, \dots, k-1$,

Since $r(G_i) \leq \frac{1}{2} r_i$ for $i = 0, \dots, k-2$ and $r(G_{k-1}) \leq \frac{1}{2} r_k$, it follows that

$$\rho(f(x), \theta_0(x)) \leq \omega r_k + C_2 \omega \sum_{i=0}^{k-1} r(G_i) \leq C_3 \omega \beta,$$

where $C_3 = \frac{1}{4} + C_2$.

Consider the case $\frac{32}{7} \omega \geq \frac{1}{2}$. Without loss of generality we may assume that $\varepsilon \geq \varepsilon_4$ for a constant ε_4 of the proof of Lemma 8, and θ_i is just the left translation satisfying $\theta_i(x_i) = f(x_i)$. Thus we can apply Lemma 13. The theorem follows in the same way. \square

6 Appendix

6.1 Application of the embedding theorem

Lemma 11. *Let $f \in I(1 + \varepsilon, B(a, r))$ and*

$$\|D_h f - I\|_{p, B(a, r)} \leq \varepsilon |B(a, r)|^{1/p}.$$

If $p > \nu$ and $f(a) = a$ then

$$\rho(f(x), x) \leq Cr(\sqrt{\varepsilon} + \varepsilon) \quad \text{for all } x \in B(a, sr)$$

with $s \in (0, 1)$. The constant C depends only on n, p , and s .

Proof. Put $B = B(a, r)$. Denote the first $2n$ coordinates of $x^{-1} \cdot f(x)$ by $\psi(x)$ and the last coordinate of $x^{-1} \cdot f(x)$ by $\chi(x)$. Estimate $\nabla_{\mathcal{L}} \psi_i(x)$ for all $i = 1, \dots, 2n$ and $\nabla_{\mathcal{L}} \chi(x)$. Clearly,

$$\|\nabla_{\mathcal{L}} \psi_i\|_{p, B} = \|\nabla_{\mathcal{L}} f_i - \nabla_{\mathcal{L}} x_i\|_{p, B} \leq \|D_h f - I\|_{p, B}, \quad i = 1, \dots, 2n.$$

The embedding theorem (see [7] for example) yields

$$|\psi_k(x)| \leq C_1 r^{1-\nu/p} \|\nabla_{\mathcal{L}} \psi_k\|_{p, B} \leq C_2 \varepsilon r \quad \text{for all } x \in \frac{s+1}{2} B, \quad k = 1, \dots, 2n.$$

We have

$$\chi(x) = f_{2n+1}(x) - x_{2n+1} + 2 \sum_{j=1}^n (x_j f_{j+n}(x) - x_{j+n} f_j(x)).$$

The contact condition $X_i f(x) \in H_{f(x)} \mathbb{H}^n$ for $i = 1, \dots, 2n$ yields

$$X_i f_{2n+1}(x) = 2 \sum_{j=1}^n f_{j+n}(x) X_i f_j(x) - f_j(x) X_i f_{j+n}(x),$$

and then we deduce that $\nabla_{\mathcal{L}} \chi(x) = 2((D_h f(x))^t + I) J \psi(x)$, where J is the $2n \times 2n$ matrix defined in (8).

Applying the embedding theorem once again, we obtain

$$|\chi(x)| \leq C_3 r^{1-\nu/p} \|\nabla_{\mathcal{L}} \chi\|_{p, \frac{s+1}{2} B} \leq C_4 r \|\psi\|_{C(\frac{s+1}{2} B)} (2 + \varepsilon) \leq C_5 r^2 \varepsilon (2 + \varepsilon) \quad \text{for all } x \in sB.$$

Hence, $\rho(f(x), x) \leq C_6 r(\sqrt{\varepsilon(2 + \varepsilon)} + \varepsilon) \leq C_7 r(\sqrt{\varepsilon} + \varepsilon)$ for all $x \in B(a, sr)$. \square

6.2 Isometries on the balls

Lemma 12. *If φ is an isometry on \mathbb{H}^n with $\rho(\varphi(x), x) \leq \varepsilon r$ for all $x \in \overline{B(a, r)} \subset \mathbb{H}^n$ with $\varepsilon < 1/2$, then $\rho(\varphi(x), x) \leq 5\varepsilon sr$ for all $x \in \overline{B(a, sr)}$, $s \geq 1$.*

Proof. Assume that $B(a, r) = B(0, 1)$. Suppose firstly that $\varphi = \iota \circ \pi_{\mathbf{a}} \circ \varphi_A$ where $\mathbf{a} = (a, \alpha) \in \mathbb{H}^n$ with $a \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}$, as well as $A \in U(n)$. If $x = 0$ then $|a| < 1/2$ and $|\alpha| \leq 1/4$. If $z = 0$ and $t = 1$ then we arrive at a contradiction:

$$1/2 \geq \rho(\varphi(0, 1), (0, 1)) = \rho((\bar{a}, -\alpha - 2)) \geq \sqrt{2 + \alpha}.$$

Thus, $\varphi = \pi_{\mathbf{a}} \circ \varphi_A$, where $\mathbf{a} = (a, \alpha) \in \mathbb{H}^n$, $a \in \mathbb{C}^n$, $\alpha \in \mathbb{R}$, and $A \in U(n)$. We have

$$x^{-1} \cdot \mathbf{a} \cdot \varphi_A x = (-z + a + Az, \alpha + 2 \operatorname{Im}\langle a, Az \rangle - 2 \operatorname{Im}\langle z, a + Az \rangle).$$

Clearly, $|\mathbf{a}| = \rho(\varphi(0), 0) \leq \varepsilon$ and $|Az - z| \leq |Az - z + a| + |a| \leq 2\varepsilon$.

We have

$$\begin{aligned} 2|\operatorname{Im}\langle 2a + Az, z \rangle| &\leq |\alpha + 2 \operatorname{Im}\langle a, Az \rangle - 2 \operatorname{Im}\langle z, Az + a \rangle| \\ &\quad + |\alpha| + 2|\operatorname{Im}\langle a, Az - z + a \rangle| \leq 4\varepsilon^2 \quad \text{for all } z \in \mathbb{C}^n, |z| \leq 1. \end{aligned}$$

Suppose that $Aa = \xi a + d$ and $(d, a) = 0$ with $\xi \in \mathbb{C}$ and $d \in \mathbb{C}^n$. Put $z = \gamma a$, $|z| \leq 1$. Then

$$|\operatorname{Im}\langle 2a + Az, z \rangle| = |\operatorname{Im}\langle 2a + \gamma(\xi a + d), \gamma a \rangle| = |a|^2 |\operatorname{Im}(2\bar{\gamma} + \xi|\gamma|^2)| \leq 2\varepsilon^2.$$

Suppose that $a \neq 0$. For $\gamma = \frac{1}{|a|}$, we infer that $|\operatorname{Im}\langle 2a + Az, z \rangle| = |\operatorname{Im} \xi| \leq 2\varepsilon^2$. For $\gamma = \frac{-i}{|a|}$,

$$2|a| \leq |a|^2 |\operatorname{Im}(2\bar{\gamma} + \xi|\gamma|^2)| + |\operatorname{Im} \xi| \leq 4\varepsilon^2.$$

Hence,

$$|\operatorname{Im}\langle Az, z \rangle| \leq |\operatorname{Im}\langle 2a + Az, z \rangle| + |\operatorname{Im}\langle 2a, z \rangle| \leq 6\varepsilon^2.$$

In the case of $a = 0$, we obviously have $|\operatorname{Im}\langle Az, z \rangle| \leq 2\varepsilon^2$.

Consider $y = \delta_s x \in B(0, s)$. We obtain

$$y^{-1} \cdot \mathbf{a} \cdot \varphi_A y = (-sz + a + sAz, \alpha + 2 \operatorname{Im}\langle a, Asz \rangle - 2 \operatorname{Im}\langle sz, a + Asz \rangle).$$

Then $|-sz + a + sAz| \leq s|Az - z| + |a| \leq (2s + 1)\varepsilon$ and

$$\begin{aligned} |\alpha + 2 \operatorname{Im}\langle a, Asz \rangle - 2 \operatorname{Im}\langle sz, a + Asz \rangle| \\ \leq |\alpha| + 2|\operatorname{Im}\langle a, Asz + sz \rangle| + 2|\operatorname{Im}\langle sz, Asz \rangle| \leq (1 + 8s + 12s^2)\varepsilon^2. \end{aligned}$$

Thus, $\rho(\pi_{\mathbf{a}} \circ \varphi_A(y), y) \leq 5s\varepsilon$.

Now, take an arbitrary ball $B(a, r)$ and suppose that $\rho(\varphi(x), x) \leq \varepsilon r$ on $B(a, r)$. The isometry $\theta = \delta_{1/r} \circ \pi_{-a} \circ \varphi \circ \pi_a \circ \delta_r$ satisfies $\rho(\theta(y), y) \leq 5s\varepsilon$ for all $y \in B(0, s)$. Inserting $x = a \cdot \delta_r y$ for $x \in B(a, sr)$, we obtain the required estimate. \square

The following lemma is obvious.

Lemma 13. *If $\rho(bx, x) \leq \varepsilon$ on $B(a, r)$ then $\rho(bx, x) < 3s\varepsilon$ on $B(a, sr)$, $s \geq 1$.*

References

- [1] *Arcozzi N., Morbidelli D.* Stability of isometric maps in the Heisenberg group, *Comment. Math. Helv.* **83** (2008), no. 1, 101–141.
- [2] *Buckley S., Koskela P., Lu G.* Boman equals John, *Proc. XVI Rolf Nevanlinna Colloquium*, 1996, 91–99.
- [3] *Buckley S. M.* Inequalities of John–Nirenberg type in doubling spaces, *J. Anal. Math.* **79** (1999), 215–240.
- [4] *Citti G., Sarti A.*, A cortical based model of perceptual completion in the roto-translation space, *Proceedings of the Workshop on Second-Order Subelliptic Equations and Applications* (Cortona, Italy, June 16–22, 2003). Potenza: Università degli Studi della Basilicata, Dipartimento di Matematica, S.I.M. Lecture Notes of Seminario Interdisciplinare di Matematica, 2004. **3**, 145–161.
- [5] *Dunford N., Schwartz J. T.*, *Linear Operators. I. General Theory*. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, New York–London, 1958.
- [6] *Friesecke G., James R. D., Müller S.*, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Commun. Pure Appl. Math.* **55** (2002), no. 11, 1461–1506.
- [7] *Garofalo N., Nhieu D.-M.*, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996), no. 10, 1081–1144.
- [8] *Gromov M.*, Carnot–Carathéodory spaces seen from within. In: *Sub-Riemannian geometry*, Birkhäuser, Basel, 1996, 79–318.
- [9] *Hladky R. K., Pauls S. D.*, Minimal surfaces in the roto-translation group with applications to a neuro-biological image completion model, *J. Math. Imaging Vision* **36** (2010), no. 1, 1–27.
- [10] *Hurri–Syrjänen R.*, The John–Nirenberg inequality and a Sobolev inequality for general domains, *J. Math. Anal. Appl.* **175** (1993), 579–587.

- [11] *Isangulova D. V.*, The class of mappings with bounded specific oscillation, and integrability of mappings with bounded distortion on Carnot groups, *Sibirsk. Mat. Zh.* **48** (2007), no. 2, 313–334; English translation in: *Siberian Math. J.* **48** (2007), no. 2, 249–267.
- [12] *Isangulova D. V.*, Local stability of mappings with bounded distortion on Heisenberg groups, *Sibirsk. Mat. Zh.* **48** (2007), no. 6, 1228–1245; English translation in: *Siberian Math. J.* **48** (2007), no. 6, 984–997
- [13] *Isangulova D. V.*, *Vodopyanov S. K.*, Coercive estimates and integral representation formulas on Carnot groups, *Eurasian Math. J.* **1** (2010), no. 3, 58–96.
- [14] *John F.*, Rotation and strain, *Commun. Pure Appl. Math.* **14** (1961), 391–413.
- [15] *John F.*, Bounds for deformations in terms of average strains. In: *Inequalities III. Proc. Sympos. (California, 1969)*, Academic, New York, 1972.
- [16] *Karmanova M.B.*, *Vodop'yanov S. K.*, Geometry of Carnot – Carathéodory spaces, differentiability, coarea and area formulas, In: *Analysis and mathematical physics (B. Gustavsson, A. Vasil'ev Ed.)*. Trends in Mathematics. Birkhäuser: Basel–Boston–Berlin, 2009, 233–335.
- [17] *Korányi A.*, *Reimann H. M.*, Foundations for the theory of quasiconformal mappings on the Heisenberg group, *Adv. Math.* **111** (1995), no. 1, 1–87.
- [18] *Landau L. D.*, *Lifshitz E. M.*, *Course of Theoretical Physics. Vol. 7. Theory of Elasticity*, Fourth Edition, Nauka, Moscow, 1986; English translation: Pergamon Press, Oxford, 1986.
- [19] *Malcev A. I.*, *Foundations of Linear Algebra*, Gostechizdat, Moscow, 1956 (Russian); English translation: W. H. Freeman & Co., San Francisco, Calif.-London, 1963.
- [20] *Nagel A.*, *Stein E. M.*, *Wainger S.*, Balls and metrics defined by vector fields I: Basic properties, *Acta Math.* **155** (1985), 103–147.
- [21] *Pansu P.*, Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un, *Acta Math.* **129** (1989), no. 1, 1–60.
- [22] *Smith W.*, *Stegenda D. A.*, Exponential integrability of the quasihyperbolic metric in Hölder domains, *Ann. Acad. Sci. Fenn. Ser. A I. Math.* **16** (1991), 345–460.
- [23] *Reshetnyak Yu. G.*, *Stability Theorems in Geometry and Analysis*, Mathematics and its Applications, 304. Kluwer Academic Publishers, Dordrecht, 1994.

- [24] *Romanovskiĭ N. N.*, Integral representations and embedding theorems for functions defined on the Heisenberg groups \mathbb{H}^n , *Algebra i Analiz* **16** (2004), no. 2, 82–119; English translation in: *St. Petersburg Math. J.* **16** (2005), no. 2, 349–375.
- [25] *Romanovskiĭ N. N.*, On the Mikhlin problem on Carnot groups, *Sibirsk. Mat. Zh.* **49** (2008), n. 1, 193–206; English translation in: *Siberian Math. J.* **49** (2008), no. 1, 155–165.
- [26] *Vodopyanov S. K.*, Closure of classes of mappings with bounded distortion on Carnot groups, *Mat. Tr.* **5** (2002), no. 2, 92–137; English translation in: *Siberian Adv. Math.* **14** (2004), no. 1, 84–125.
- [27] *Vodopyanov S. K.*, Geometry of Carnot–Carathéodory Spaces and Differentiability of Mappings, In: *The interaction of analysis and geometry*, *Contemp. Math.*, 424, Amer. Math. Soc., Providence, RI, 2007, 247–301.
- [28] *Vodopyanov S. K.*, *Isangulova D. V.*, Sharp geometric rigidity of isometries on Heisenberg groups, *Dokl. Akad. Nauk.* **482** (2008), no. 5, 583–588; English translation in: *Doklady Mathematics* **77** (2008), no. 3, 432–437.